Nonstandard Fourier transformation and Feynman path integral

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Abstract. We construct an infinitesimal Fourier transformation for the space of functionals. We extend $\mathbf{R}$ to ${}^\star({}^*\mathbf{R})$ under the base of nonstandard analysis for the construction. The domain of a functional is the set of all internal functions from a *-finite lattice to a ${}^{\star\circ*}$-finite lattice with double meanings. Considering a ${}^{\star\circ*}$-finite lattice with double meanings, we find how to treat the domain for a functional in our theory of Fourier transformation.

Recently many kinds of geometric invariants are defined on manifolds and they are used for studying low dimensional manifolds, for example, Donaldson's invariant, Chern-Simon's invariant and so on. They are originally defined as Feynman path integrals in physics. The Feynman path integral is in a sense an integral of a functional on an infinite dimensional space of functions. We would like to study the Feynman path integral and the originally defined invariants. For the purpose, we would be sure that it is necessary to construct a theory of Fourier transformation on the space of functionals. For it, as the later argument, we would need many stages of infinitesimals and infinites, that is, we need to put a concept of stage on the field of real numbers. We use nonstandard analysis to develop a theory of Fourier transformation on the space of functionals.

Historically, for the theories of Fourier transformations in nonstandard analysis, in 1972, Luxemburg([L2]) developed a theory of Fourier series with *-finite summation on the basis of nonstandard analysis. The basic idea of his approach is to replace the usual ∞ of the summation to an infinite natural number N. He approximated the Fourier transformation on the unit circle by the Fourier transformation on the group of Nth roots of unity.

In 1988, Kinoshita([K]) defined a discrete Fourier transformation for each even *-finite number $H(\in {}^*\mathbf{R})$: $(F\varphi)(p) = \sum_{-\frac{H^2}{2}\le z<\frac{H^2}{2}} \frac{1}{H}\exp(-2\pi i p\frac{1}{H}z)\varphi(\frac{1}{H}z)$, called "infinitesimal Fourier transformation". He developed a theory for the infinitesimal Fourier transformation and studied the distribution space deeply, and proved the same properties hold as usual Fourier transformation of $L^2(\mathbf{R})$.

In 1989, Gordon([G]) independently defined a generic, discrete Fourier transformation for each infinitesimal Δ and *-finite number M, defined by $(F_{\Delta,M}\,\varphi)(p) = \sum_{-M\le z\le M}\Delta\exp(-2\pi i p\Delta z)\varphi(\Delta z)$. He studied under which condition the discrete Fourier transformation $F_{\Delta,M}$ approximates the usual Fourier transformation $\mathcal{F}$ for $L^2(\mathbf{R})$. His proposed condition is (A') of his notation : let Δ be an infinitely small and M an infinitely large natural number such that $M\cdot\Delta$ is infinitely large. He showed that under the condition (A') the standard part of $F_{\Delta,M}\,\varphi$ approximates the usual $\mathcal{F}\varphi$ for $\varphi\in L^2(\mathbf{R})$. One of the different points between Kinoshita's and Gordon's is that there is the term $\Delta\exp(-2\pi i p\Delta M)\varphi(\Delta M)$ in the summation of their two definitions or not. We mention that both definitions are same for the standard part of the dicrete Fourier transformation for $\varphi \in L^2(\mathbf{R})$ and Kinoshita's definition satisfies the condition (A') for an even infinite number H if $\Delta = \frac{1}{H}$, $M = \frac{H^2}{2}$.

We shall extend their theory of Fourier transformation for the space of functions to a thery of Fourier transformation for the space of functionals. For the purpose of this, we shall represent a space of functions from $\mathbf{R}$ to $\mathbf{R}$ as a space of functions from a set of lattices in an infinite interval $\left[-\frac{H}{2}, \frac{H}{2}\right)$ to a set of lattices in an infinite interval $\left[-\frac{H'}{2}, \frac{H'}{2}\right)$. We consider what H' is to treat any function from $\mathbf{R}$ to $\mathbf{R}$. If we put a function $a(x) = x^n (n \in \mathbf{Z}^+)$, we need that $\frac{H'}{2}$ is greater than $\left(\frac{H}{2}\right)^n$, and if we choose a function $a(x) = e^x$, we need that $\frac{H'}{2}$ is greater than $e^{\frac{H}{2}}$. If we choose any infinite number, there exists a function of which image is beyond the infinite number. Since we treat all functions from $\mathbf{R}$ to $\mathbf{R}$, we need to put $\frac{H'}{2}$ as an infinite number greater than any infinite number of ${}^*\mathbf{R}$. Hence we make $\left[-\frac{H'}{2}, \frac{H'}{2}\right)$ not in ${}^*\mathbf{R}$ but in ${}^\star({}^*\mathbf{R})$, where ${}^\star({}^*\mathbf{R})$ is a double extension of $\mathbf{R}$, that is, H' is an infinite number in ${}^\star({}^*\mathbf{R})$. First we shall develop an infinitesimal Fourier transformation theory for the space of functionals, and secondly we calculate standard two examples for our infinitesimal Fourier transformation.

Formulation.

To explain our infinitesimal Fourier transformation for the space of functionals, we introduce Kinoshita's infinitesimal Fourier transformation for the space of functions. We fix an infinite set Λ and an ultrafilter F of Λ so that F includes the Fréchet filter $F_0(\Lambda)$. We remark that the set of natural numbers is naturally embedded in Λ. Let H be an even infinite number where the definition being even is the following : if H is written as $[(H_\lambda, \lambda \in \Lambda)]$ then $\{\lambda \in \Lambda \mid H_\lambda \text{ is even}\} \in F$. Let ε be $\frac{1}{H}$, that is, if ε is $[(\varepsilon_\lambda, \lambda \in \Lambda)]$ then ε_λ is $\frac{1}{H_\lambda}$. Then we shall define a lattice space $\mathbf{L}$, a sublattice space L and a space of functions $R(L)$:

$$\mathbf{L} := \varepsilon\, {}^*\mathbf{Z} = \{\varepsilon z \mid z \in {}^*\mathbf{Z}\},$$

$$L := \left\{\varepsilon z \;\middle|\; z \in {}^*\mathbf{Z},\ -\tfrac{H}{2} \le \varepsilon z < \tfrac{H}{2}\right\} = \left\{[(\varepsilon_\lambda z_\lambda),\ \lambda \in \Lambda] \;\middle|\; \varepsilon_\lambda z_\lambda \in L_\lambda\right\} \quad (\subset \mathbf{L})$$

$$R(L) := \{\varphi \mid \varphi \text{ is an internal function from } L \text{ to } {}^*\mathbf{C}\} = \{[(\varphi_\lambda, \lambda \in \Lambda)] \mid \varphi_\lambda \text{ is a function from } L_\lambda \text{ to } \mathbf{C}\},$$

where $L_\lambda := \left\{\varepsilon_\lambda z_\lambda \;\middle|\; z_\lambda \in \mathbf{Z},\ -\frac{H_\lambda}{2} \le \varepsilon_\lambda z_\lambda < \frac{H_\lambda}{2}\right\}$.

Kinoshita([K]) introduced an infinitesimal delta function $\delta(x) (\in R(L))$ and an infinitesimal Fourier transformation on $R(L)$. From now on, functions in $R(L)$ are extended to periodic functions on $\mathbf{L}$ with the period H and we denote them by the same notations. For $\varphi (\in R(L))$, the infinitesimal Fourier transformation $F\varphi$, the inverse infinitesimal Fourier transformation $\overline{F}\varphi$, and the convolution of $\varphi, \psi (\in R(L))$ are defined as follows :

$$\delta(x) := \begin{cases} H & (x = 0), \\ 0 & (x \neq 0), \end{cases}$$

$$(F\varphi)(p) := \sum_{x \in L} \varepsilon \exp(-2\pi i p x)\, \varphi(x), \quad (\overline{F}\varphi)(p) := \sum_{x \in L} \varepsilon \exp(2\pi i p x)\, \varphi(x),$$

$$(\varphi * \psi)(x) := \sum_{y \in L} \varepsilon \varphi(x - y) \psi(y).$$

He obtained the following equalities as same as the usual Fourier analysis :

$\delta = F1 = \overline{F}1,\ F$ is unitary, $F^4 = 1,\ \overline{F}F = F\overline{F} = 1,$
$\varphi * \delta = \delta * \varphi = \varphi,\ \varphi * \psi = \psi * \varphi,$
$F(\varphi * \psi) = (F\varphi)(F\psi),\ F(\varphi\psi) = (F\varphi) * (F\psi),$
$\overline{F}(\varphi * \psi) = (\overline{F}\varphi)(\overline{F}\psi),\ \overline{F}(\varphi\psi) = (\overline{F}\varphi) * (\overline{F}\psi).$

On the other hand, we obtain the following theorem from his result and an elementary calculation :

Theorem 1 For an internal function with two variables $f : L \times L \to {}^*\mathbf{C}$ and $g(\in R(L))$,

$$F_x\left(\sum_{y\in L} \varepsilon f(x-y,y)g(y)\right)(p) = \{F_y(F_u(f(u,y))(p)) * F_y(g(y))\}(p),$$

where $F_x,\ F_y,\ F_u$ are Fourier transformations for $x,\ y,\ u,$ and $*$ is the convolution for the variable paired with y by the Fourier transformation.

Proof. By the above Kinoshita's result, $F(\varphi\psi) = (F\varphi) * (F\psi)$. We use it and obtain the following :

$$F_x\left(\sum_{y\in L} \varepsilon f(x-y,y)g(y)\right)(p) = \sum_{x,y\in L} \varepsilon \exp(-2\pi ipx)\varepsilon f(x-y,y)g(y)$$
$$= \sum_{y,u\in L} \varepsilon^2 \exp(-2\pi ip(y+u))f(u,y)g(y)\ (u := x-y)$$
$$= \sum_{y\in L}\left(\varepsilon \exp(-2\pi ipy)\left(\sum_{u\in L}\varepsilon\exp(-2\pi pu)f(u,y)\right)g(y)\right)$$
$$= F_y(F_u(f(u,y))(p)\cdot g(y))(p) = \{F_y(F_u(f(u,y))(p)) * F_y(g(y))\}(p).$$

We explain our infinitesimal Fourier transformation for the space of functionals. Let $\Lambda_1,\ \Lambda_2$ be infinite sets, and let $F_1,\ F_2$ be ultrafilters on $\Lambda_1,\ \Lambda_2$ so that $F_0(\Lambda_1) \subset F_1,\ F_0(\Lambda_2) \subset F_2$ where $F_0(\Lambda_1)$ and $F_0(\Lambda_2)$ are Fréchet filters for $\Lambda_1,\ \Lambda_2$. We denote the ultraproduct of a set S_1 for F_1 by ${}^\star S_1$ and the ultraproduct of a set S_2 for F_2 by ${}^*S_2$. Then an element of ${}^\star({}^*S)$ is written as $[(s_\lambda),\ \lambda \in \Lambda_1]$, where $s_\lambda \in {}^*S,\ s_\lambda = [(s_{\lambda\mu}),\ \mu \in \Lambda_2],\ s_{\lambda\mu} \in S$. We use the same notation $[\quad]$ for representing the equivarence classes for both F_1 and F_2, and we write the images of $s_1 \in S_1,\ s_2 \in S_2$ by the natural elementary embeddings $\star : S_1 \to {}^\star S_1,\ * : S_2 \to {}^*S_2$, as ${}^\star s_1,\ {}^*s_2$, if there is no confusion.

An infinite number in ${}^\star({}^*\mathbf{R})$ is defined to be greater than any element in ${}^*\mathbf{R}$. We remark that an infinite number in ${}^*\mathbf{R}$ is not infinite in ${}^\star({}^*\mathbf{R})$, that is, the word "an infinite number in ${}^\star({}^*\mathbf{R})$" has double meamings. An infinitesimal number in ${}^\star({}^*\mathbf{R})$ is also defined to be nonzero and whose absolute value is less than each positive number in ${}^*\mathbf{R}$.

Definition 2 Let $H(\in {}^*\mathbf{Z}),\ H'(\in {}^\star({}^*\mathbf{Z}))$ be even infinite numbers which are written as $[(H_\mu),\ \mu \in \Lambda_2],\ [(H'_\lambda),\ \lambda \in \Lambda_1]\ (H'_\lambda = [(H'_{\lambda\mu}),\ \mu \in \Lambda_2])$, and let $\varepsilon(\in {}^*\mathbf{R}),\ \varepsilon'(\in {}^\star({}^*\mathbf{R}))$ be infinitesimals satifying $\varepsilon H = 1,\ \varepsilon' H' = 1$. We define as follows :

$$\mathbf{L} := \varepsilon\, {}^*\mathbf{Z} = \{\varepsilon z \mid z \in {}^*\mathbf{Z}\},\ \ \mathbf{L}' := \varepsilon'\, {}^\star({}^*\mathbf{Z}) = \{\varepsilon' z' \mid z' \in {}^\star({}^*\mathbf{Z})\},$$
$$L := \left\{\varepsilon z\ \middle|\ z \in {}^*\mathbf{Z},\ -\tfrac{H}{2} \le \varepsilon z < \tfrac{H}{2}\right\}(\subset \mathbf{L}),$$
$$L' := \left\{\varepsilon' z'\ \middle|\ z' \in {}^\star({}^*\mathbf{Z}),\ -\tfrac{H'}{2} \le \varepsilon' z' < \tfrac{H'}{2}\right\}(\subset \mathbf{L}').$$

Here L is an ultraproduct of lattices

$$L_\mu := \left\{\varepsilon_\mu z_\mu\ \middle|\ z_\mu \in \mathbf{Z},\ -\tfrac{H_\mu}{2} \le \varepsilon_\mu z_\mu < \tfrac{H_\mu}{2}\right\}\ (\mu \in \Lambda_2)$$

in $\mathbf{R}$, and L' is also an ultraproduct of lattices

$$L'_\lambda := \left\{\varepsilon'_\lambda z'_\lambda\ \middle|\ z'_\lambda \in {}^*\mathbf{Z},\ -\tfrac{H'_\lambda}{2} \le \varepsilon'_\lambda z'_\lambda < \tfrac{H'_\lambda}{2}\right\}\ (\lambda \in \Lambda_1)$$

in ${}^*\mathbf{R}$ that is an ultraproduct of

$L'_{\lambda\mu} := \left\{ \varepsilon'_{\lambda\mu} z'_{\lambda\mu} \,\middle|\, z'_{\lambda\mu} \in \mathbf{Z},\, -\frac{H'_{\lambda\mu}}{2} \leq \varepsilon'_{\lambda\mu} z'_{\lambda\mu} < \frac{H'_{\lambda\mu}}{2} \right\}$ $(\mu \in \Lambda_2)$.

We define a latticed space of functions X as follows,

$X := \{a \,|\, a$ is an internal function with double meamings, from $\star\,(L)$ to $L'\}$

$= \{[(a_\lambda),\, \lambda \in \Lambda_1] \,|\, a_\lambda$ is an internal function from L to $L'_\lambda\}$,

where $a_\lambda : L \to L'_\lambda$ is $a_\lambda = [(a_{\lambda\mu}),\, \mu \in \Lambda_2],\ a_{\lambda\mu} : L_\mu \to L'_{\lambda\mu}$.

We define three equivarence relations $\sim_H$, $\sim_{\star(H)}$ and $\sim_{H'}$ on $\mathbf{L}$, $\star(\mathbf{L})$ and $\mathbf{L}'$:

$x \sim_H y \Longleftrightarrow x - y \in H\,{}^*\mathbf{Z},\ \ x \sim_{\star(H)} y \Longleftrightarrow x - y \in \star(H)\,{}^\star({}^*\mathbf{Z})$,

$x \sim_{H'} y \Longleftrightarrow x - y \in H'\,{}^\star({}^*\mathbf{Z})$.

Then we identify $\mathbf{L}/\sim_H$, $\star(\mathbf{L})/\sim_{\star(H)}$ and $\mathbf{L}'/\sim_{H'}$ as L, $\star(L)$ and L'. Since $\star(L)$ is identified with L, the set $\star(\mathbf{L})/\sim_{\star(H)}$ is identified with $\mathbf{L}/\sim_H$. Furthermore we represent X as the following internal set :

$\{a \,|\, a$ is an internal function with double meamings, from $\star(\mathbf{L})/\sim_{\star(H)}$ to $\mathbf{L}'/\sim_{H'}$ $\}$.

We use the same notation as a function from $\star(L)$ to L' to represent a function in the above internal set. We define the space A of functionals as follows :

$A := \{f \,|\, f$ is an internal function with double meamings, from X to ${}^\star({}^*\mathbf{C})\}$.

Then f is written as $f = [(f_\lambda),\, \lambda \in \Lambda_1]$, f_λ is an internal function from the set $\{a_\lambda \,|\, a_\lambda$ is an internal function from L to $L'_\lambda\}$ to ${}^*\mathbf{C}$, and f_λ is written as $f_\lambda = [(f_{\lambda\mu}),\, \mu \in \Lambda_2],\ f_{\lambda\mu} : \{a_{\lambda\mu} : L_\mu \to L'_{\lambda\mu}\} \to \mathbf{C}$.

We define an infinitesimal delta function $\delta(a)(\in A)$, an infinitesimal Fourier transformation of $f(\in A)$, an inverse infinitesimal Fourier transformation of f and a convolution of f, $g(\in A)$, by the following :

Definition 3

$$\delta(a) := \begin{cases} (H')^{({}^\star H)^2} & (a = 0), \\ 0 & (a \neq 0), \end{cases}$$

$\varepsilon_0 := (H')^{-({}^\star H)^2} \in {}^\star({}^*\mathbf{R})$,

$(Ff)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k) b(k)\right) f(a)$,

$(\overline{F}f)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i \sum_{k \in L} a(k) b(k)\right) f(a)$,

$(f * g)(a) := \sum_{a' \in X} \varepsilon_0 f(a - a') g(a')$.

We define an inner product on A : $(f, g) := \sum_{b \in X} \varepsilon_0 \overline{f(b)} g(b)$, where $\overline{f(b)}$ is the complex conjugate of $f(b)$. Then we obtain the following theorem :

Theorem 4

(1) $\delta = F1 = \overline{F}1$, (2) F is unitary, $F^4 = 1, \overline{F}F = F\overline{F} = 1$,

(3) $f * \delta = \delta * f = f$, (4) $f * g = g * f$,

(5) $F(f * g) = (Ff)(Fg)$, (6) $\overline{F}(f * g) = (\overline{F}f)(\overline{F}g)$,

(7) $F(fg) = (Ff) * (Fg)$, (8) $\overline{F}(fg) = (\overline{F}f) * (\overline{F}g)$.

We define two types of infinitesimal divided differences. Let f and a be elements of A and X respectively and let $b(\in X)$ be an internal function whose image is in ${}^\star({}^*\mathbf{Z}) \cap L'$. We remark that $\varepsilon' b$ is an element of X.

Definition 5

$(D_{+,b} f)(a) := \frac{f(a + \varepsilon' b) - f(a)}{\varepsilon'}$, $(D_{-,b} f)(a) := \frac{f(a) - f(a - \varepsilon' b)}{\varepsilon'}$.

Let $\lambda_b(a) := \frac{\exp(2\pi i \varepsilon' ab) - 1}{\varepsilon'}$, $\overline{\lambda}_b(a) := \frac{\exp(-2\pi i \varepsilon' ab) - 1}{\varepsilon'}$. Then we obtain the following theorem corresponding to Kinoshita's result for the relationship between the infinitesimal Fourier transformation and the infinitesimal divided differences :

Theorem 6

(1) $(F(D_{+,b}f))(a) = \lambda_b(a)(Ff)(a)$, (2) $(F(D_{-,b}f))(a) = -\overline{\lambda}_b(a)(Ff)(a)$,
(3) $(F(\lambda_b f))(a) = -(D_{-,b}(Ff))(a)$, (4) $(F(\overline{\lambda_b} f))(a) = (D_{+,b}(Ff))(a)$,
(5) $(D_{+,b}(\overline{F}f))(a) = (\overline{F}(\lambda_b f))(a)$, (6) $(D_{-,b}(\overline{F}f))(a) = -(\overline{F}(\overline{\lambda_b}f))(a)$,
(7) $\lambda_b(a) = 2\pi i \left(\frac{\sin(\pi\varepsilon' ab)}{\pi\varepsilon'}\right)\exp(\pi i\varepsilon' ab)$.

Replacing the definitions of L', F, $\overline{F}$, δ in Definition 2 and Definition 3 by the following, we shall define another type of infinitesimal Fourier transformation. The different point is only the definition of an inner product of the space of functions X. In Definition 3, the inner product of a, $b(\in X)$ is $\sum_{k\in L} a(k)b(k)$, and in the following definition, it is ${}^{\star}\varepsilon\sum_{k\in L} a(k)b(k)$.

Definition 7

$L' := \left\{\varepsilon' z' \;\middle|\; z' \in {}^{\star}({}^{*}\mathbf{Z}),\, -{}^{\star}H\frac{H'}{2} \leq \varepsilon' z' < {}^{\star}H\frac{H'}{2}\right\}$,

$$\delta(a) := \begin{cases} ({}^{\star}HH')^{({}^{\star}H)^2} & (a = 0), \\ 0 & (a \neq 0), \end{cases}$$

$(Ff)(b) := \sum_{a\in X}\varepsilon_0 \exp\left(-2\pi i\,{}^{\star}\varepsilon\sum_{k\in L} a(k)b(k)\right) f(a)$,
$(\overline{F}f)(b) := \sum_{a\in X}\varepsilon_0 \exp\left(2\pi i\,{}^{\star}\varepsilon\sum_{k\in L} a(k)b(k)\right) f(a)$.

In this case, we obtain the same theorems as Theorem 4 and Theorem 6, and the following theorem corresponding to Theorem 1 :

Theorem 8 For an internal function with two variables $f : X \times X \to {}^{\star}({}^{*}\mathbf{C})$ and $g(\in A)$,

$F_a\left(\sum_{b\in X}\varepsilon_0 f(a-b,b)g(b)\right)(d) = \{F_b(F_c(f(c,b))(d)) * F_b(g(b))\}(d)$,
where F_a, F_b, F_c are Fourier transformations for a, b, c, and $*$ is the convolution for the variable pairing with b by the Fourier transformation.

Examples.

We calculate two examples of the infinitesimal Fourier transformation for the space A of functionals. Let $\star\circ * : \mathbf{R} \to {}^{\star}({}^{*}\mathbf{R})$ be the natural elementary embedding and let $\mathbf{st}(c)$ for $c \in {}^{\star}({}^{*}\mathbf{R})$ be the standard part of c with respect to the natural elementary embedding $\star\circ *$. The first is for $\exp(\pi i\sum_{k\in L}{}^{\star}\varepsilon a^2(k))$ and the second is for $\exp(-\pi\sum_{k\in L}{}^{\star}\varepsilon a^2(k))$. We denote the two functionals by $f(a), g(a)$. If there is an L^2-function $\alpha(t)$ on $\mathbf{R}$ for $a(k)$ so that $a(k) = \star(({}^{*}\alpha)(k))$, then $\mathbf{st}(f(a)) = \exp\left(\pi i\int_{-\infty}^{\infty}\alpha^2(t)dt\right)$, and $\mathbf{st}(g(a)) = \exp\left(-\pi\int_{-\infty}^{\infty}\alpha^2(t)dt\right)$. Then we obtain the following results :

Example 1. $(Ff)(b) = C_1\overline{f(b)}$, where $C_1 = \sum_{a\in X}\varepsilon_0\exp(i\,{}^{\star}\varepsilon\pi\sum_{k\in L}(a(k)^2)$, and it is constant,

Example 2. $(Fg)(b) = C_2(b)g(b)$, where $C_2(b) = \sum_{a\in X}\varepsilon_0\exp(-{}^{\star}\varepsilon\pi\sum_{k\in L}(a(k)+ib(k)^2)$, and if b is a finite valued function then it satisfies that $\mathrm{st}\left(\mathrm{st}\left(\frac{C_2(b)}{C_2(0)}\right)\right) = 1$.

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